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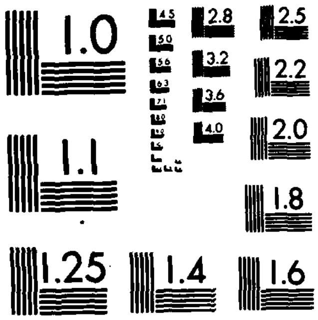

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A new class of tests that can be viewed as a generalization of Neyman's optimal $C(\alpha)$ test is introduced. An optimally robust test which maximizes the asymptotic power within this class, subject to a bounded influence function, is selected. It is shown that it is equivalent to a certain asymptotically minimax test proposed by Wang (1981). Finally, some numerical results on the asymptotic behaviour of robust $C(\alpha)$-type tests under several errors' and carriers' distributions are discussed.

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1. INTRODUCTION

Let $\{(x_i, y_i) : i=1, \ldots, n\}$ be a sequence of independent identical distributed random variables such that

$$y_i = x_i^T \theta + e_i \ , \qquad i=1, \ldots, n \ , \tag{1.1}$$

where y_i is the ith observation, $x_i \varepsilon \mathbb{R}^p$ is the ith row (written as a column vector) of the design matrix, $\theta \ \varepsilon \ \Theta \subset \mathbb{R}^p$ a p-vector of unknown parameters and $e_i \varepsilon \mathbb{R}$ the ith error. Suppose that e_i is independent of x_i and is distributed according to a normal $N(0,\sigma^2)$. Moreover, denote by $K(x)$ the distribution of the x's with respect to some σ-finite measure μ and by $F_\theta(x,y)$ the joint distribution of (x_i, y_i).

In the classical parametric approach, there are several procedures one can use for subhypothesis testing: the F-test, which is equivalent to the likelihood ratio test, the Wald test and the $C(\alpha)$ test. These tests are asymptotically equivalent and, when the errors are normally distributed, they are optimal; see, for instance, Cox and Hinkley (1974). However, these test procedures suffer similar robustness problems as the least squares estimators. Although they are moderately robust with respect to the level, they do lose power rapidly in the presence of small departures from the normality assumption on the errors; cf. Hampel (1973, 1978), Schrader and Hettmansperger (1980). Robust alternatives to the F-test and the Wald test have been investigated recently by Schrader and McKean (1977), Schrader and Hettmansperger (1980) and Ronchetti (1982a, 1982b, 1984).

In this paper we focus on the class of $C(\alpha)$ tests. In section 2 we define a new class of tests which generalizes the optimal $C(\alpha)$ tests introduced by Neyman (1958, 1979). In section 3 we investigate their robustness properties by means of the influence function and we compute their asymptotic power. This allows us to select the optimally robust $C(\alpha)$-type test, that is a test which maximizes the asymptotic power within this new class (efficiency condition), subject to a bounded influence function of the test statistic (robustness requirement). In section 4 we show the equivalence between our optimally robust $C(\alpha)$-type test and an asymptotically minimax test introduced by Wang (1981). This points out the strong relation between the minimax approach and our approach to robust testing. Finally, section 5 presents some numerical results on the asymptotic behaviour of $C(\alpha)$-type tests under several distributions of the errors and of the carriers.

2. C(α)-TYPE TESTS

Consider the regression model (1.1). Though $C(\alpha)$ tests can be defined for testing more than one linear hypothesis on θ (see Bühler and Puri, 1966), we focus here on the generalization of Neyman's original definition (see Neyman, 1958), that is we shall introduce a new class of procedures for testing

$$H_0 : \theta^{(p)} = 0 \tag{2.1}$$

where $\theta^{(j)}$ denotes the jth component of the vector θ. For a given p-vector x, we denote by $x_{(1)}$ its "nuisance part", that is $x_{(1)} = (x^{(1)},\ldots,x^{(p-1)})^T$ and by $x_{(2)}$ its last component $x^{(p)}$. Moreover, let $x = (x_{(1)}^T, 0)^T$. We denote by $M_{(11)}$, $M_{(12)}$, $M_{(21)}$, $M_{(22)}$ the submatrices of a $(p-1)\times 1$ partition of a $p\times p$ matrix M. Note that $M_{(22)}$ is simply m_{pp}, the (p,p) component of M.

The class of tests we shall define is based on a function

$$\eta : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R} \ , \ (x,r) \to \eta(x,r) \ ,$$

which satisfies the following conditions:

(2.ETA1) (i) $\eta(x,\cdot)$ is continuous and odd for all $x \in \mathbb{R}^p$,

(ii) $\eta(x,r) \geq 0$ for all $x \in \mathbb{R}$, $r \in \mathbb{R}^+$

(2.ETA2) $\eta(x,\cdot)$ is differentiable on $\mathbb{R}\setminus D(x;\eta)$ for all $x \in \mathbb{R}^p$, where $\mathcal{D}(x;\eta)$ is a finite set.

Let $\eta'(x,r) := (\partial/\partial r)\eta(x,r)$ if $x \in \mathbb{R}^p$, $r \in \mathbb{R}\backslash \mathcal{D}(x;\eta)$

$:= 0$ otherwise,

and assume:

$\sup_r |\eta'(x,r)| < \infty$ for all $x \in \mathbb{R}^p$.

We shall also assume the following regularity condition:

(2.ETA3) $M := E\eta'(x,r)xx^T$ exists and is nonsingular.

Definition 2.1 The class of $C(\alpha)$-type tests for linear models is defined by means of the test statistics

$$\begin{aligned} Z_n(\tilde{\theta};\eta) &:= n^{-\frac{1}{2}} \sum_{i=1}^{n} \eta(x_i,(y_i-x_i^T\tilde{\theta})/\sigma)\cdot(U^{-1}x_i)^{(p)} \\ &= n^{-\frac{1}{2}} u_{pp}^{-1} \sum_{i=1}^{n} \eta(x_i,(y_i-x_i^T\tilde{\theta})/\sigma)\cdot[x_i^{(p)}-U_{(21)}\cdot U_{(11)}^{-1}(x_i)_{(1)}] \end{aligned} \tag{2.2}$$

where η satisfies the conditions (2. ETA1), (2.ETA2) and U is the lower triangular matrix with positive diagonal elements such that $UU^T = M$. "Large" values of Z_n are significant.

Remark 1. $C(\alpha)$ tests were introduced by Neyman (1958) for a general parametric model and extended to the robust testing problem by Wang (1981, Remarks 2 and 3, p. 1100) who was able to derive an (asymptotically minimax) robust version of the optimal $C(\alpha)$ test. We shall discuss the connections with this test in section 4. If we put $\eta(x,r) = -\phi'(r)/\phi(r) = r$ in (2.2), where ϕ is the standard normal density (the density of the error's distribution), Z_n becomes the test statistic of the optimal $C(\alpha)$ test obtained by Neyman (1958, p. 228). In this sense the tests defined by (2.2) can be called $C(\alpha)$-type tests.

Remark 2. The test statistic Z_n depends on the unknown nuisance parameters $\theta^{(1)},\ldots,\theta^{(p-1)}$ and on the scale parameter σ . In section 3 we shall discuss the properties of a studentized version of Z_n . From now on we assume for simplicity $\sigma = 1$.

Remark 3. In order to determine asymptotic critical regions of the test, we shall compute the asymptotic distribution of Z_n in section 3.

3. OPTIMALLY ROBUST $C(\alpha)$-TYPE TESTS

In this section we investigate the robustness properties of $C(\alpha)$-type tests by means of the influence function and we select the optimally robust test in this class. This is a test that maximizes the asymptotic power within the class of $C(\alpha)$-type test, subject to a bounded influence function.

The notion of influence function for estimators was introduced by Hampel (1968, 1974). It is essentially the first derivative of an estimator viewed as functional, and describes the normalized influence of an infinitesimal observation on the estimator. Formally, suppose the estimator T_n can be written as functional T of the empirical distribution function $F^{(n)}$, $T_n = T(F^{(n)})$. Then the influence function of T at F is given by

$$IF(z;T,F) = \lim_{\varepsilon \to 0}(T((1-\varepsilon)F+\varepsilon\delta_z) - T(F))/\varepsilon \quad , \qquad (3.1)$$

where δ_z is the distribution that puts mass 1 at the point z. It is then clear that, from a robustness point of view, boundedness is a desirable property of IF.

The concept of influence function has been extended to tests by Rousseeuw and the author, and independently by Lambert; see Ronchetti (1979, 1982a); Rousseeuw and Ronchetti (1979, 1981); Hampel, Ronchetti, Rousseeuw, Stahel (1984); Lambert (1981). It turns out that the influence function defined on the test statistic is proportional to the influence of an infinitesimal observation on the level and on the power (or, as in Lamber (1981), on the P-value) of the test. Therefore, a test statistic with a bounded influence function guarantees robustness of validity and robustness of efficiency for the test.

Let us now compute the influence function of our $C(\alpha)$-type tests. Let Z be the functional corresponding to the test statistic Z_n, that is

$$Z(F) = \int \eta(x,y-x^T\tilde{\theta})\cdot(U^{-1}x)^{(p)}dF(x,y) .$$

Then $Z(F^{(n)}) = n^{-\frac{1}{2}}Z_n$, where $F^{(n)}$ is the empirical distribution function.

<u>Proposition 3.1</u> Let $F_{\tilde{\theta}}$ be the model distribution under the null hypothesis. If (2.ETA1) and (2.ETA2) hold, the influence function of Z is given by

$$IF(x,y;Z,F_{\tilde{\theta}}) = \eta(x,y-x^T\tilde{\theta})\cdot(U^{-1}x)^{(p)} \tag{3.2}$$

<u>Proof.</u> The result follows easily applying the definition (3.1) to Z and noting that by (2.ETA1) (i) $E\eta(x,r)x = 0$.

From (3.2) we see that Neyman's optimal $C(\alpha)$ test $(\eta(x,r) = r)$, though asymptotically efficient, has an unbounded influence function. Our goal will be to compromise between efficiency and robustness within the class of $C(\alpha)$-type tests by finding a test that maximizes the asymptotic power, under a bound on the influence function. Let us now compute the asymptotic power of a $C(\alpha)$-type test defined by η .

<u>Proposition 3.2</u> Besides (2.ETA1), (2.ETA2) and (2.ETA3) assume

(3.AS3) $E\eta^2(x,r)\cdot\|x\|^2 < \infty$,

where $\|\cdot\|$ denotes the Euclidean norm. Put $\lambda_p := E\eta^2(x,r)\cdot|(U^{-1}x)^{(p)}|^2$.
Then, under the sequence of alternatives

$$H_{(n)} : \theta = \tilde{\theta} + n^{-\frac{1}{2}}\Delta \quad , \tag{3.3}$$

where $\Delta = (0,\ldots,0,\Delta^{(p)})^T$,
the test statistic Z_n has asymptotically a normal distribution with mean $u_{pp}\cdot\Delta^{(p)}$ and variance λ_p. Moreover, the asymptotic power of the $C(\alpha)$-type test at the level α defined by Z_n is given by

$$1 - \Phi(\Phi^{-1}(1-\alpha) - \Delta^{(p)}\cdot u_{pp}/\lambda_p^{\frac{1}{2}}) \quad ,$$

where Φ is the cumulative standard normal.

Proof. It suffices to compute $E_\theta Z_n$ under the sequence (3.3). Define:

$$\xi(\theta) := \int\eta(x,y-x^T\tilde{\theta})\cdot(U^{-1}x)^{(p)}\cdot f_\theta(x,y)\cdot d\mu(x)dy \quad ,$$

where $f_\theta(x,y) = \phi(y-x^T\theta)\cdot k(x)$. Then, for $j = 1,\ldots,p-1$, we have

$$\begin{aligned}[(\partial/\partial\theta^{(j)})\xi(\theta)]_{\tilde{\theta}} = & -\int\eta'(x,y-x^T\tilde{\theta})x^{(j)}\cdot(U^{-1}x)^{(p)}dF_{\tilde{\theta}}(x,y) \\ & + \int\eta(x,y-x^T\tilde{\theta})\cdot(U^{-1}x)^{(p)}\cdot[(\partial/\partial\theta^{(j)})f_\theta(x,y)]_{\tilde{\theta}}d\mu(x)dy \ .\end{aligned} \tag{3.4}$$

Using

$$(U^{-1}x)^{(p)} = u_{pp}^{-1}\cdot(-U_{(21)}U_{(11)}^{-1}x_{(1)}+x^{(p)}) \ , \tag{3.5}$$

we get

$$\begin{aligned} &E\eta'(x,r)x_{(1)}^{T}\cdot(U^{-1}x)^{(p)} = \\ &u_{pp}^{-1}\cdot(-U_{(21)}\cdot U_{(11)}^{-1}\cdot E\eta'(x,r)x_{(1)}x_{(1)}^{T} + E\eta'(x,r)x_{(1)}^{T}x^{(p)}) = \\ &u_{pp}^{-1}\cdot(-U_{(21)}\cdot U_{(11)}^{-1}\cdot U_{(11)}\cdot U_{(11)}^{T} + M_{(12)}^{T}) = 0 \ , \end{aligned} \tag{3.6}$$

and the first term of the right hand side of (3.4) vanishes. In a similar way one can prove that also the second term of the right hand side of (3.4) equals 0. Moreover,

$$\begin{aligned} &[(\partial/\partial\theta^{(p)})\xi(\theta)]_{\tilde{\theta}} = \\ &\int\eta(x,y-x^{T}\tilde{\theta})\cdot(U^{-1}x)^{(p)}\cdot[(\partial/\partial\theta^{(p)})f_{\theta}(x,y)]_{\tilde{\theta}}d\mu(x)dy = \\ &E\eta(x,r)\cdot(U^{-1}x)^{(p)}\cdot r\cdot x^{(p)} = \end{aligned}$$

and by integrating by parts,

$$\begin{aligned} &E\eta'(x,r)\cdot(U^{-1}x)^{(p)}\cdot x^{(p)} = \\ &u_{pp}^{-1}\cdot(-U_{(21)}\cdot U_{(11)}^{-1}\cdot M_{(12)} + m_{pp}) = u_{pp}^{-1}(u_{pp}^{2}) = u_{pp} \ . \end{aligned} \tag{3.7}$$

Finally, denoting by a dot the differentiation with respect to θ , we obtain

$$(\dot{\xi}(\tilde{\theta}))^{(j)} = 0 \quad , \quad \text{for } j = 1,\dots,p-1 \tag{3.8}$$

$$(\dot{\xi}(\tilde{\theta}))^{(p)} = u_{pp} \tag{3.9}$$

and by (2.ETA1) (i)

$$\xi(\tilde{\theta}) = 0 \ .$$

Therefore, using a Taylor expansion we get

$$\begin{aligned} E_\theta Z_n &= n^{1/2}\xi(\theta) = n^{1/2}\dot{\xi}^T(\tilde{\theta})\cdot(\theta-\tilde{\theta}) + o(||\theta-\tilde{\theta}||) \\ &= u_{pp}\Delta^{(p)} + o(n^{-1/2}) \ . \end{aligned}$$

This completes the proof.

<u>Remark 1.</u> The test defined by the test statistic Z_n depends on the unknown nuisance parameter $\tilde{\theta}$. By techniques similar to those used by Wang (1981) one can show that the result of Proposition 3.2 holds if we substitute $\tilde{\theta}$ by a suitable ($n^{1/2}$-consistent) estimate T_n (see Wang, 1981, p. 1099). Moreover, the result of Proposition 3.1 still holds. To see this, suppose that the influence function of T exists and define

$$Z(F) = \int \eta(x, y-x^T\cdot T(F))\cdot(U^{-1}x)^{(p)}dF(x,y) \ .$$

Then,

$$\begin{aligned} IF(x,y;Z,F_{\tilde\theta}) &= [(\partial/\partial\varepsilon)Z((1-\varepsilon)F_\theta + \varepsilon\delta_{(x,y)})]_{\varepsilon=0} \qquad (3.10) \\ &= \eta(x,y-x^T\tilde\theta)\cdot(U^{-1}x)^{(p)} + \int\eta(s,v-s^T\tilde\theta)\cdot(U^{-1}s)^{(p)}dF_{\tilde\theta}(s,v) \\ &\quad -(\int\eta'(s,v-s^T\tilde\theta)\cdot(U^{-1}s)^{(p)}\cdot\tilde{s}^T dF_{\tilde\theta}(s,v))\cdot IF(x,y;T\ ,F_{\tilde\theta}) \ . \end{aligned}$$

Now the second term of the right hand side of (3.10) is equal 0 in view of (3.5) and (2.ETA1) (i), and so is the third term by (3.6).

The next Proposition gives the optimally robust $C(\alpha)$-type test.

Proposition 3.3 Under (2.ETA1), (2.ETA2), (2.ETA3), (3.AS3), the test which maximizes the asymptotic power within the class of $C(\alpha)$-type tests, subject to a bounded influence function, is defined by

$$\eta_0(x,r) = |z^{(p)}|^{-1}\psi_c(r|z^{(p)}|) = \psi_{c/|z^{(p)}|}(r) \ ,$$

where $z = U_0^{-1}x$, U_0 is the lower triangular matrix defined implicitly by the matrix equation

$$E(2\Phi(c/|z^{(p)}|)-1)zz^T = I \ ,$$

$\psi_c(t) = \min(c,\max(t,-c))$ is the Huber ψ-function, and c is a given positive constant depending on the bound on the influence function.

According to Proposition 3.1 and 3.2 we have to maximize u_{pp}^2/λ_p , subject to a bound on the right hand side of (3.2). The same problem has been solved in Ronchetti (1984; Theorem 4.3) in connection with the class of so-called τ-tests. Therefore we refer to that paper for the complete proof; cf. also Ronchetti (1982a, Theorem 3).

4. CONNECTION WITH AN ASYMPTOTICALLY MINIMAX TEST

In this section we describe the connections between the optimally robust $C(\alpha)$-type test and a test introduced by Wang (1981). Wang studies the testing problem using minimax techniques. He considers the following situation.

Suppose we are given a parametric model $\{F_\theta : \theta \in \Theta \subset \mathbb{R}^p\}$ and suppose we want to test a hypothesis on one component of the parameter θ, say $\theta^{(p)} = 0$, the other $(p-1)$ components being nuisance parameters. Then, using the technique of shrinking neighbourhoods, Wang is able to derive an asymptotically minimax test in the case, where the model distribution is indexed by nuisance parameters. Let us write this result in the situation of the linear model. Define:

$$r := y - x^T\tilde{\theta} \quad , \quad a := (a^{(1)}, \ldots, a^{(p)})^T \; ,$$

$$d(x, a_{(1)}) := x^{(p)} + a_{(1)}^T x_{(1)} = x^{(p)} + \Sigma_{j=1}^{p-1} a^{(j)} x^{(j)} \; ,$$

$$\Lambda(a_{(1)}, \tilde{\theta}) = \Lambda^*(x, y; a_{(1)}, \tilde{\theta}) := r \cdot d(x, a_{(1)}) \; . \tag{4.1}$$

For a given $\varepsilon > 0$, $\delta_1 > 0$, define $V_0(a_{(1)}, \tilde{\theta})$ and $V_1(a_{(1)}, \tilde{\theta})$ implicitly by means of the equations

$$E[\max\{(\Lambda(a_{(1)}, \tilde{\theta}) - V_1(a_{(1)}, \tilde{\theta})), 0\}] = \varepsilon/\delta_1 \tag{4.2}$$

$$E[\max\{(V_0(a_{(1)}, \tilde{\theta}) - \Lambda(a_{(1)}, \tilde{\theta})), 0\} + \Lambda(a_{(1)}, \tilde{\theta})] = \varepsilon/\delta_1 \; . \tag{4.3}$$

Moreover, let $a_{(1)} = \alpha_{(1)}(\tilde{\theta})$ be the solution to the equation

$$E([\Lambda(a_{(1)}, \tilde{\theta})]_{V_0(a_{(1)}, \tilde{\theta})}^{V_1(a_{(1)}, \tilde{\theta})} \cdot r \cdot x_{(1)}) = 0 \; , \tag{4.4}$$

and define

$$v^2(\tilde{\theta}) := E([\Lambda(\alpha_{(1)},\tilde{\theta})]_{V_0(\alpha_{(1)},\tilde{\theta})}^{V_1(\alpha_{(1)},\tilde{\theta})})^2 \tag{4.5}$$

<u>Proposition 4.1</u> (Wang, 1981, p. 1099, p. 1104) The test defined by the critical region

$$\{Y_n(\tilde{\theta}) \geq \Phi^{-1}(1-\alpha) + \varepsilon V_1(\alpha_{(1)}(\tilde{\theta}),\tilde{\theta})/v(\tilde{\theta})\} \quad , \tag{4.6}$$

where

$$Y_n(\tilde{\theta}) := n^{-\frac{1}{2}}\cdot(v(\tilde{\theta}))^{-1}\cdot \sum_{i=1}^{n} [\Lambda^*(x_i,y_i;\alpha_{(1)}(\tilde{\theta}),\tilde{\theta})]_{V_0(\alpha_{(1)},\tilde{\theta})}^{V_1(\alpha_{(1)},\tilde{\theta})} \ ,$$

is an asymptotically minimax test at level α .

The test defined by (4.6) will be called Wang test. In order to have a better performance at the model, Wang (1981, p. 1105) proposes a modification of the test (4.6) and defines a test by means of the following critical region

$$\{Y_n(\tilde{\theta}) \geq \Phi^{-1}(1-\alpha)\} \tag{4.7}$$

The test defined by (4.7) will be called modified Wang test. Then we have the following result.

<u>Proposition 4.2</u> The modified Wang test is a $C(\alpha)$-type test. It is equivalent to the optimally robust $C(\alpha)$-type test defined by η_s (given by Proposition 3.3).

<u>Proof.</u> We prove that $Y_n(\tilde{\theta}) = \lambda_p^{-\frac{1}{2}}(\eta_0)\cdot z_n(\tilde{\theta};\eta_0)$. First, consider equation (4.4). Using (4.1) we have

$$0 = E([\Lambda]_{V_0}^{V_0}\cdot r\cdot x_{(1)}) = E([r\cdot d]_{V_0}^{V_1}\cdot r\cdot x_{(1)})$$
$$= \int d(x,\alpha_{(1)})\cdot x_{(1)}\cdot(\int [r]_{V_0/|d|}^{V_1/|d|}\cdot r d\Phi(r))\,dK(x) \;,$$

and (4.4) becomes

$$\int(\Phi(V_1/|d(x,\alpha_{(1)})|)-\Phi(V_0/|d(x,\alpha_{(1)})|))\cdot d(x,\alpha_{(1)})x_{(1)}\,dK(x) = 0 \;. \tag{4.8}$$

Now, combining (4.2) and (4.3) and noting that $E\Lambda = 0$, we obtain

$$\int(\Lambda-V_1)\cdot 1_{\{\Lambda\ge V_1\}} + \int(\Lambda-V_0)\cdot 1_{\{\Lambda\le V_0\}} = 0 \;,$$

and performing these integrations we get finally by (4.8)

$$V_0 = -V_1 \quad (=: -V) \;. \tag{4.9}$$

For a given positive constant c , define η_0 and U_0 as in Proposition 3.3. Moreover, choose

$$V = c\cdot(U_0)_{pp} \;. \tag{4.10}$$

Then

$$\alpha_{(1)}^{T} = -(U_0)_{(21)} \cdot ((U_0)_{(11)})^{-1} \tag{4.11}$$

and

$$\begin{aligned} d(x,\alpha_{(1)}) &= x^{(p)} - (U_0)_{(21)} \cdot ((U_0)_{(11)})^{-1} x_{(1)} \\ &= (U_0^{-1}x)^{(p)} \cdot (U_0)_{pp} \ . \end{aligned} \tag{4.12}$$

(The left member of (4.8) becomes

$$\begin{aligned} &E[(2\Phi(c/|(U_0^{-1}x)^{(p)}|)-1) \cdot (U_0^{-1}x)^{(p)} \cdot (U_0)_{pp} \cdot x_{(1)}] \\ &\quad = [E\eta_0'(x,r) \cdot (U_0^{-1}x)^{(p)} \cdot x_{(1)}] \cdot (U_0)_{pp} = 0 \quad \text{by (3.6) .)} \end{aligned}$$

Moreover, by (4.5), (4.9), (4.10) and (4.12) we have

$$\begin{aligned} v^2(\tilde{\theta}) &= E([r \cdot d]_{-V}^{+V})^2 = E(d^2(x;\alpha_{(1)}) \cdot \psi_{V/|d|}^2(r)) \\ &= (U_0)_{pp}^2 \cdot E(\psi_{c/|z^{(p)}|}^2(r) \cdot |z^{(p)}|^2) \\ &= (U_0)_{pp}^2 \cdot \lambda_p(\eta_0) \ , \end{aligned} \tag{4.13}$$

where

$$z^{(p)} = (U_0^{-1}x)^{(p)} \quad \text{and} \quad \lambda_p(\eta_0) = E\eta_0^2(x,r)\cdot|z^{(p)}|^2 \ .$$

Finally we get

$$\begin{aligned}
Y_n(\tilde{\theta}) &= n^{-1/2}\cdot((U_0)_{pp}\cdot\lambda_p^{1/2}(\eta_0))^{-1}\cdot\Sigma_{i=1}^{n}[r\cdot d]_{-V}^{+V} \\
&= n^{-1/2}\cdot\lambda_p^{-1/2}(\eta_0)\cdot\Sigma_{i=1}^{n}((U_0)_{pp})^{-1}\cdot\psi_{c(U_0)_{pp}}(r\cdot z^{(p)}\cdot(U_0)_{pp}) \\
&= n^{-1/2}\cdot\lambda_p^{-1/2}(\eta_0)\cdot\Sigma_{i=1}^{n}\psi_c(r\cdot z^{(p)}) \\
&= n^{-1/2}\cdot\lambda_p^{-1/2}(\eta_0)\cdot\Sigma_{i=1}^{n}\psi_{c/|z^{(p)}|}(r)\cdot z^{(p)} \\
&= \lambda_p^{-1/2}(\eta_0)\cdot Z_n(\tilde{\theta};\eta_0) \ .
\end{aligned}$$

This completes the proof.

Proposition 4.3 The Wang test is always less efficient at the model than the modified Wang test.

Proof. The modified Wang procedure has the same asymptotic power as the optimal robust $C(\alpha)$-type test defined by η_0 . Therefore, the square of its efficacy equals $(U_0)_{pp}^2/\lambda_p(\eta_0)$.

On the other hand, the square of the efficacy of the Wang test is given by the formula (see Wang, 1981, p. 1104)

$$s^2(\tilde{\theta}) = (v(\tilde{\theta}) - (\varepsilon/\delta_1)\cdot(-V)/v(\tilde{\theta}))^2 .$$

Thus, the relative efficiency of the Wang test with respect to the modified Wang test can be computed as

$$\text{eff}\{\text{Wang test, modified Wang test}\} = s^2(\tilde{\theta})/((U_0)^2_{pp}/\lambda_p(\eta_0))$$

hence, using (4.10) and (4.13)

$$= [\lambda_p(\eta_0)/(U_0)^2_{pp}] \cdot$$

$$[(U_0)_{pp}\cdot\lambda_p^{1/2}(\eta_0)+(\varepsilon/\delta_1)\cdot c\cdot(U_0)_{pp}/((U_0)_{pp}\cdot\lambda_p^{1/2}(\eta_0))]^2$$

$$= (\lambda_p(\eta_0) + (\varepsilon/\delta_1)\cdot c/(U_0)_{pp})^2 .$$

Using (4.2) we get finally

$$\text{eff}\{\text{Wang test, modified Want test}\}$$

$$= 1-c^2E[(|z^{(p)}|/c)\cdot\Phi'(c/|z^{(p)}|) - \Phi(-c/|z^{(p)}|)] < 1 .$$

This completes the proof.

5. ASYMPTOTIC BEHAVIOUR

In this section we present some numerical results on the asymptotic behaviour of $C(\alpha)$-type tests under several distributions. In the exposition we apply the methods which have been used by Maronna, Bustos and Yohai (1979) for comparing different regression estimators. We consider simple regresssion:

$$y = x^T\theta + e ,$$

where $x = (1, x^{(2)})^T$, $\theta = (\theta^{(1)} , \theta^{(2)})^T$.
We want to test the hypothesis

$$H_0 : \theta^{(2)} = 0 .$$

Let G and K be the distributions of e and $x^{(2)}$, respectively. For $x^{(2)}$ and e we choose the following contaminated normal distributions

$$(1-\varepsilon)\Phi(\cdot) + \varepsilon\Phi(\cdot/s)$$

with the following parameters

$$\begin{array}{lllll} \varepsilon : & 0 & 0.1 & 0.1 & 0.05 \\ s : & 1 & 3 & 5 & 10 \end{array} .$$

We want to compare the asymptotic behaviour of the optimally robust $C(\alpha)$-type test (see Proposition 3.3)

$$\eta_0(x,r) = \psi_{c_1}/|z^{(2)}|(r)$$

with that of another $C(\alpha)$-type test with bounded influence function, namely

$$\eta_M(x,r) = w_{c_2}(|z^{(2)}|)\psi_{c_3}(r) \ , \qquad (5.1)$$

where $W_c(|t|) = \psi_c(t)/t = \min(1,c/|t|)$. The $C(\alpha)$-type test defined by (5.1) is the solution of the optimality problem considered in Proposition 3.3 within the restricted class of functions η satisfying $\eta(x,r) = w(x)\cdot\psi(r)$; see Ronchetti (1982b). The subscript "M" reminds that η_M is a function of Mallows' form.

In order to investigate the behaviour of a $C(\alpha)$-type test defined by a "redescending" η-function, we consider also the following procedure:

$$\eta_{MT}(x,r) = w_{c_4}(|z^{(2)}|)\cdot\psi(r;c_5,\kappa^*,c_6,A_5,B_5) \ ,$$

where ψ is defined by

$$\begin{aligned}
\psi(r;c_5,\kappa^*,c_6,A_5,B_5) &= r && \text{if } 0 \le |r| \le c_6 \\
&= \alpha\cdot\tanh[\beta(c_5-|r|)]\text{sign}(r) && c_6 \le |r| \le c_5 \\
&= 0 && c_5 \le |r| \ ,
\end{aligned}$$

$\alpha = (A_5(\kappa^*-1))^{1/2}$, $\beta = ((\kappa^*-1)B_5^2/A_5)^{1/2}/2$. The function ψ defines the socalled "hyperbolic tangent estimator" for location; see Hampel, Rousseeuw, Ronchetti (1981). Note that c_6, A_5, B_5 are computed implicitly in terms of c_5 and κ^* .

For each test we compute the standardized sensitivities at the normal model

$$u_{22}^{-1} \cdot \sup_{x,r} |\eta(x,r)| \cdot |z^{(2)}|$$

and the efficacies

$$u_{22}^2/\lambda_2$$

under several distributions.

Note that the efficacy of the tests defined by η_M and η_{MT} factorizes; in these cases we have

$$u_{22}^2/\lambda_2 = DX \cdot DR ,$$

where DX depends only on w_c and K and $DR = B^2/A$, with $A = E\psi_c^2$, $B = E\psi_c'$. The constants are chosen so that all the tests have the same asymptotic efficiency, when $x^{(2)} \sim N(0,1)$ and $e \sim N(0,1)$.

Table 1 describes the calibration as well as the standardized sensitivities of the tests. From Table 2 one can obtain the following conclusions:

1) η_{MT} is better than η_M for all distributions under consideration;

2) η_0 is better than η_{MT} when the distribution of the errors has moderate tails;

3) η_0 has the better standardized sensitivities (computed at the normal model).

Table 3 gives the factors for the η_M- and η_{MT}-test.

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Table 1: Calibration and standardized sensitivities at $x^{(2)} \sim N(0,1)$, $e \sim N(0,1)$

TEST	CONSTANTS	DX	DR	EFF	u_{22}	A	B	ST.SENS.
η_M	$c_2=1.80$ $c_3=1.58$	.98	.97	.95	.900	.809	.886	3.16
η_{MT}	$c_4=1.80$ $c_5=4.68$ $\kappa^*=4.50$ $c_6=1.70$ $A_5=.84$ $B_5=.90$	.98	.97	.95	.910	.842	.905	3.37
η_S	$c_1=2.87$	-	-	.95	.930	-	-	2.87

Table 2: Asymptotic efficacies

K	TEST	G: $\varepsilon=0$, $s=1$	G: $\varepsilon=.1$, $s=3$	G: $\varepsilon=.1$, $s=5$	G: $\varepsilon=.05$, $s=10$	ST.SENS.
$\varepsilon=0$, $s=1$	η_M	.95	.75	.71	.80	3.16
	η_{MT}	.95	.77	.77	.87	3.37
	η_0	.95	.78	.72	.78	2.87
$\varepsilon=.1$, $s=3$	η_M	1.52	1.21	1.13	1.28	2.70
	η_{MT}	1.53	1.23	1.23	1.40	2.91
	η_0	1.54	1.27	1.16	1.25	2.34
$\varepsilon=.1$, $s=5$	η_M	2.44	1.94	1.81	2.04	2.33
	η_{MT}	2.44	1.97	1.97	2.24	2.51
	η_0	2.51	2.06	1.86	2.00	1.94
$\varepsilon=.05$, $s=10$	η_M	2.99	2.38	2.23	2.51	2.23
	η_{MT}	3.00	2.42	2.42	2.76	2.38
	η_0	3.24	2.61	2.32	2.50	1.85

Table 3: Factorization for η_M- and η_{MT}-test

	$\varepsilon=0$ $s=1$	$\varepsilon=.1$ $s=3$	$\varepsilon=.1$ $s=5$	$\varepsilon=.05$ $s=10$
		DR		
η_M	.970	.771	.722	.814
η_{MT}	.973	.785	.785	.894
		DX		
	.979	1.569	2.512	3.087